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INFLUENCE OF GENERAL IMPERFECTIONS IN AXIALLY LOADED CYLINDRICAL SHELLS

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Abstract—The buckling of an axially loaded cylindrical shell is considered when imperfection components corresponding to all of the classical buckling modes are taken into consideration. The analysis represents an extension of Koiter's axisymmetric solution and in the asymptotic sense due to Koiter the imperfections considered are as general as possible. The results obtained reveal many interesting aspects of shell buckling which arize for various imperfection forms. The buckling behaviour which results is associated with both bifurcation and limit point critical states.

INTRODUCTION

The buckling behaviour and consequently the imperfection sensitivity of shell structures has provided a basis for much theoretical and experimental research. Historically, one of the key theoretical contributions was that due to Koiter[1] in which he determined the influence of axisymmetric initial imperfections on the critical load of an axially loaded cylindrical shell. He recognized that axisymmetric imperfections did not yield a complete explanation of the imperfection sensitivity for this problem and in a later paper[2] he discussed a single non-axisymmetric imperfection and suggested that it would be worthwhile to consider more general imperfection shapes. Further, the papers by Chilver and Johns[3], Hutchinson[4], Arbocz and Babcock[5], as well as Johns[6] indicate that multiple mode behaviour, which will be induced by general imperfections, should lead to some interesting aspects of imperfection sensitivity.

The present paper considers the first order influence of completely general imperfections within the context of Koiter's asymptotic approach, this being accomplished through an extension of the analysis in [1]. It is found that, within the limits of the asymptotic analysis, there are only five imperfection parameters which are of importance in the determination of the critical load. These five quantities are in turn completely specified by simple relationships which involve all of the imperfection components corresponding to the classical buckling modes.

POTENTIAL ENERGY

In order that the presentation be complete it is necessary to synopsize Koiter's[1] analysis. The elastic energy, obtained by Koiter through an extension of Love's shell theory, is

$$
\begin{aligned}
\bar{V} = {} & 1/4\frac{Gh}{1-\nu}\int_0^{l/R}\mathrm{d}x\int_0^{2\pi}\mathrm{d}\theta\{4u'^2+4(\dot{v}+w)^2+8\nu u'(\dot{v}+w) \\
& +2(1-\nu)(\dot{u}+v')^2+k[w''^2+(\ddot{w}-\dot{v})^2+2\nu w''(\ddot{w}-\dot{v}) \\
& +2(1-\nu)(\dot{w}'-v')^2]\} \\
& +1/4\frac{Gh}{1-\nu}\frac{1}{R}\int_0^{l/R}\mathrm{d}x\int_0^{2\pi}\mathrm{d}\theta\{4u'(u'^2+v'^2+w'^2) \\
& +4(\dot{v}+w)[\dot{u}^2+(\dot{v}+w)^2+(\dot{w}-v)^2] \\
& +4\nu u'[\dot{u}^2+(\dot{v}+w)^2+(\dot{w}-v)^2]+4\nu(\dot{v}+w)(u'^2+v'^2+w'^2) \\
& +4(1-\nu)(\dot{u}+v')[u'\dot{u}+v'(\dot{v}+w)+w'(\dot{w}-v)]\} \\
& +1/4\frac{Gh}{1-\nu}\frac{1}{R^2}\int_0^{l/R}\mathrm{d}x\int_0^{2\pi}\mathrm{d}\theta\{(u'^2+v'^2+w'^2)^2 \\
& +[\dot{u}^2+(\dot{v}+w)^2+(\dot{w}-v)^2]^2 \\
& +2\nu(u'^2+v'^2+w'^2)[\dot{u}^2+(\dot{v}+w)^2+(\dot{w}-v)^2] \\
& +2(1-\nu)[u'\dot{u}+v'(\dot{v}+w)+w'(\dot{w}-v)]^2\},
\end{aligned} \tag{1}
$$

where G is the shear modulus, h the thickness, ν Poisson's ratio, R the radius of the shell, l the length, x is the non-dimensional axial coordinate, θ is the polar angle, u, v, w are the axial, circumferential and radial displacement components respectively. The letter k is defined by $k = h^2/3R^2$. Differentiation with respect to x is indicated by a prime and with respect to θ by a dot. The load parameter is defined as $\lambda = N/Eh$ where N is the axial load per unit circumference acting on the cylinder. Thus the energy due to the applied load is

$$-\bar{W} = 2\lambda Gh(1+\nu)R\int_0^{2\pi}\int_0^{l/R} u'\,\mathrm{d}x\,\mathrm{d}\theta. \tag{2}$$

The fundamental state is found to be

$$U = -\lambda Rx, \quad V = 0, \quad W = \nu\lambda R. \tag{3}$$

Consequently, the displacements take the form

$$U + u = -\lambda Rx + u, \quad V + v = v, \quad W + w = \lambda\nu R + w. \tag{4}$$

The energy expressions for a transition of the system from the fundamental state to an adjacent state are obtained by substituting eqns (4) into eqns (1) and (2). The energy is then expanded and all terms of degree greater than first in U, V and W are neglected. Therefore, the energy can be expressed in the form

$$P^\lambda[u] = P_2{}^0[u] + \lambda P_2'[u] + P_3{}^0[u] + \lambda P_3'[u] + P_4{}^0[u] + P_4'[u] \tag{5}$$

where

$$P_2{}^0[u] = \int_0^{l/R}\int_0^{2\pi}\{4u'^2 + 4(\dot{v}+w)^2 + 8\nu u'(\dot{v}+w) + 2(1-\nu)(\dot{u}+v')^2$$
$$+ k[w''^2 + (\ddot{w}-\dot{v})^2 + 2\nu w''(\ddot{w}-\dot{v}) + 2(1-\nu)(\dot{w}'-v')^2]\}\,\mathrm{d}\theta\,\mathrm{d}x,$$

$$P_2'[u] = \int_0^{l/R}\int_0^{2\pi}\{(-12+4\nu^2)u'^2 + 8\nu(\dot{v}+w)^2 - 8\nu(1-\nu)u'(\dot{v}+w)$$
$$-4(1-\nu)\dot{u}^2 - 4(1-\nu)v'^2 - 4(1-\nu^2)\dot{u}v' - 4(1-\nu^2)w'^2\}\,\mathrm{d}\theta\,\mathrm{d}x,$$

$$P_3{}^0[u] = \frac{1}{R}\int_0^{l/R}\int_0^{2\pi}\{4u'(u'^2+v'^2+w'^2) + 4(\dot{v}+w)[\dot{u}^2 + (\dot{v}+w)^2 + (\dot{w}-v)^2]$$
$$+4\nu u'[\dot{u}^2 + (\dot{v}+w)^2 + (\dot{w}+v)^2] + 4\nu(\dot{v}+w)(u'^2+v'^2+w'^2)$$
$$+4(1-\nu)(\dot{u}+v')[u'\dot{u} + v'(\dot{v}+w) + w'(\dot{w}-v)]\}\,\mathrm{d}\theta\,\mathrm{d}x,$$

$$P_3'[u] = \frac{1}{R}\int_0^{l/R}\int_0^{2\pi}\{4[-u' + \nu^2(\dot{v}+w)](u'^2+v'^2+w'^2)$$
$$+4\nu(-u'+\dot{v}+w)[\dot{u}^2 + (\dot{v}+w)^2 + (\dot{w}-v)^2]$$
$$-4(1-\nu)(\dot{u}-\nu v')[u'\dot{u} + v'(\dot{v}+w) + w'(\dot{w}-v)]\}\,\mathrm{d}\theta\,\mathrm{d}x,$$

$$P_4{}^0[u] = \frac{1}{R^2}\int_0^{l/R}\int_0^{2\pi}\{(u'^2+v'^2+w'^2)^2 + [\dot{u}^2 + (\dot{v}+w)^2 + (\dot{w}-v)^2]^2$$
$$+2\nu(u'^2+v'^2+w'^2)[\dot{u}^2 + (\dot{v}+w)^2 + (\dot{w}-v)^2]$$
$$+2(1-\nu)[u'\dot{u} + v'(\dot{v}+w) + w'(\dot{w}-v)]^2\}\,\mathrm{d}\theta\,\mathrm{d}x,$$

$$P_4'[u] = 0.$$

CLASSICAL CRITICAL LOAD

The classical critical load is defined by the smallest eigenvalue of the linearized problem. After some simplification the eigenvalues of the linearized problem are found to be

$$\lambda = \frac{p^2}{(p^2+n^2)^2} + \frac{k}{4(1-\nu)^2}\frac{(p^2+n^2)^2}{p^2} \tag{6}$$

where p and n are the axial and circumferential wave numbers. The former of these is defined as $p = i\pi R/l$. The quantities i and n range over the positive integers as well as zero. If p is treated as a continuous variable, the minimum value of λ occurs when

$$p_{n1,2} = \frac{1}{2}\left[\frac{4(1-\nu^2)}{k}\right]^{1/4} \pm \left[\frac{1}{2}\sqrt{\frac{(1-\nu^2)}{k}} - n^2\right]^{1/2}, \tag{7}$$

where n takes all integer values, including zero which yield real values for p. Also p_{n1}, p_{n2} represent the roots of the quadratic equation for p. The minimum value of λ thus obtained is

$$\lambda_1 = \left(\frac{k}{1-\nu^2}\right)^{1/2}. \tag{8}$$

Also, since $n = p = 0$ represents a translation in the axial direction, it is ignored. Therefore, when $n = 0$, the only value of p considered is

$$p_0 = \left[\frac{4(1-\nu^2)}{k}\right]^{1/4}. \tag{9}$$

The assumption that p is continuous implies that the shell is infinitely long. This inturn requires that the boundary conditions be replaced by a periodicity requirement.

The eigenfunctions corresponding to the classical critical load are

$$\begin{aligned} w = {} & a_0 \sin p_0x + b_0 \cos p_0x + \sum_{n=1}^{m} (a_{n1} \sin p_{n1}x \cos n\theta + b_{n1} \sin p_{n1}x \sin n\theta \\ & + c_{n1} \cos p_{n1}x \cos n\theta + d_{n1} \cos p_{n1}x \sin n\theta + a_{n2} \sin p_{n2}x \cos n\theta \\ & + b_{n2} \sin p_{n2}x \sin n\theta + c_{n2} \cos p_{n2}x \cos n\theta + d_{n2} \cos p_{n2}x \sin n\theta), \end{aligned} \tag{10}$$

$$\begin{aligned} u = {} & \frac{\nu}{p_0}(a_0 \cos p_0x - b_0 \sin p_0x) + \sum_{n=1}^{m} \left\{ p_{n1} \frac{(\nu p_{n1}^2 - n^2)}{(p_{n1}^2 + n^2)^2} (a_{n1} \cos p_{n1}x \cos n\theta + b_{n1} \cos p_{n1}x \sin n\theta \right. \\ & - c_{n1} \sin p_{n1}x \cos n\theta - d_{n1} \sin p_{n1}x \sin n\theta) \\ & + p_{n2} \frac{(\nu p_{n2}^2 - n^2)}{(p_{n2}^2 + n^2)^2} (a_{n2} \cos p_{n2}x \cos n\theta + b_{n2} \cos p_{n2}x \sin n\theta - c_{n2} \sin p_{n2}x \cos n\theta \\ & \left. - d_{n2} \sin p_{n2}x \sin n\theta) \right\}, \end{aligned} \tag{11}$$

$$\begin{aligned} v = {} & \sum_{n=1}^{m} \left\{ \frac{n[(2+\nu)p_{n1}^2 + n^2]}{(p_{n1}^2 + n^2)^2} (-a_{n1} \sin p_{n1}x \sin n\theta + b_{n1} \sin p_{n1}x \cos n\theta - c_{n1} \cos p_{n1}x \sin n\theta \right. \\ & + d_{n1} \cos p_{n1}x \cos n\theta) + \frac{n[(2+\nu)p_{n2}^2 + n^2]}{(p_{n2}^2 + n^2)^2} (-a_{n2} \sin p_{n2}x \sin n\theta + b_{n2} \sin p_{n2}x \cos n\theta \\ & \left. - c_{n2} \cos p_{n2}x \sin n\theta + d_{n2} \cos p_{n2}x \cos n\theta) \right\}. \end{aligned} \tag{12}$$

where m is the largest value of n which leads to real values of $p_{n1,2}$.

POTENTIAL ENERGY—FIRST APPROXIMATION

Substitution of the above expressions for u, v and w into eqns (5) yields an algebraic expression which represents a first approximation to the potential energy in the context of a Koiter style analysis. This expression, after retention of only the predominant terms, is

$$\begin{aligned} P^{\lambda}[u_1] = {} & \frac{2\pi l}{R}(1-\nu^2)\left\{ -R(\lambda - \lambda_1)\left[2p_0^2(a_0^2 + b_0^2) \right.\right. \\ & \left. + \sum_{n=1}^{m} p_{n1}^2(a_{n1}^2 + b_{n1}^2 + c_{n1}^2 + d_{n1}^2) + p_{n2}^2(a_{n2}^2 + b_{n2}^2 + c_{n2}^2 + d_{n2}^2) \right] \\ & + 3\sum_{n=1}^{m} n^2[b_0(-a_{n1}a_{n2} - b_{n1}b_{n2} + c_{n1}c_{n2} + d_{n1}d_{n2}) \\ & \left. + a_0(a_{n1}c_{n2} + a_{n2}c_{n1} + b_{n1}d_{n2} + b_{n2}d_{n1})] \right\}. \end{aligned} \tag{13}$$

The first order approximation to the potential energy due to the presence of initial radial imperfections in the shell is obtained by substituting the same expressions for u, v and w into

$$\lambda Q_1'[u] = 8\lambda \int_0^{l/R}\int_0^{2\pi} w_0[(1-\nu^2)w'' + \nu(\nu u' + \dot{v} + w)]\,d\theta\,dx, \tag{14}$$

where $w_0 = w_0(x, \theta)$ is the initial radial imperfection. This yields

$$\begin{aligned}\lambda Q_1'[u_1] = \frac{-2\pi l}{R^2}(1-\nu^2)8R\lambda\Big\{&p_0^2(A_0a_0 + B_0b_0)\\ &+\sum_{n=1}^{m} p_{n1}^2(A_{n1}a_{n1} + B_{n1}b_{n1} + C_{n1}c_{n1} + D_{n1}d_{n1})\\ &+ p_{n2}^2(A_{n2}a_{n2} + B_{n2}b_{n2} + C_{n2}c_{n2} + D_{n2}d_{n2})\Big\},\end{aligned} \tag{15}$$

where

$$A_0 = \frac{R}{2\pi l}\int_0^{l/R}\int_0^{2\pi} w_0 \sin p_0x\,d\theta\,dx,$$

$$A_{n1} = \frac{R}{2\pi l}\int_0^{l/R}\int_0^{2\pi} w_0 \sin p_{n1}x \cos n\theta\,d\theta\,dx,$$

and so on. A_0, $A_{n1}, \ldots$ will be called the imperfection parameters.

EQUILIBRIUM EQUATIONS

The algebraic equilibrium equations are obtained by setting the first partial derivatives of the energy expression with respect to a_0, b_0, $a_{n1}, \ldots$ equal to zero. Consequently, the following set of equations is obtained

$$-8R\lambda p_0^2A_0 - 4R(\lambda-\lambda_1)p_0^2a_0 + 3\sum_{n=1}^{m} n^2(a_{n1}c_{n2} + a_{n2}c_{n1} + b_{n1}d_{n2} + b_{n2}d_{n1}) = 0, \tag{16}$$

$$-8R\lambda p_0^2B_0 - 4R(\lambda-\lambda_1)p_0^2b_0 + 3\sum_{n=1}^{m} n^2(-a_{n1}a_{n2} - b_{n1}b_{n2} + c_{n1}c_{n2} + d_{n1}d_{n2}) = 0, \tag{17}$$

$$-8R\lambda p_{n1}^2A_{n1} - 2R(\lambda-\lambda_1)p_{n1}^2a_{n1} + 3n^2(-b_0a_{n2} + a_0c_{n2}) = 0, \tag{18}$$

$$-8R\lambda p_{n2}^2A_{n2} - 2R(\lambda-\lambda_1)p_{n2}^2a_{n2} + 3n^2(-b_0a_{n1} + a_0c_{n1}) = 0, \tag{19}$$

$$-8R\lambda p_{n1}^2C_{n1} - 2R(\lambda-\lambda_1)p_{n1}^2c_{n1} + 3n^2(b_0c_{n2} + a_0a_{n2}) = 0, \tag{20}$$

$$-8R\lambda p_{n2}^2C_{n2} - 2R(\lambda-\lambda_1)p_{n2}^2c_{n2} + 3n^2(b_0c_{n1} + a_0a_{n1}) = 0. \tag{21}$$

A further set of four equations is obtained for $b_{n1,2}$, $d_{n1,2}$ by replacing $A_{n1,2}$, $C_{n1,2}$ and $a_{n1,2}$, $c_{n1,2}$ with $B_{n1,2}$, $D_{n1,2}$ and $b_{n1,2}$, $d_{n1,2}$, respectively. These last eight equations for $a_{n1,2}$, $b_{n1,2}$, $c_{n1,2}$ and $d_{n1,2}$ are then repeated for all values of n ranging from 1 to m. There are, therefore, $2+8m$ simultaneous equilibrium equations. At this stage, Koiter proceeds when only A_0 and B_0 are non-zero. This leads to an investigation of the influence of axisymmetric initial imperfections. On the other hand, Koiter points out that every set of four equations is only coupled to the first two equations. Further, every set of four equations is linear if a_0, b_0 are viewed as parameters. The determination of $a_{n1,2}$, $b_{n1,2}, \ldots$ is straight-forward and yields the results

$$a_{n1} = (-8R\lambda)\frac{\{2R(\lambda-\lambda_1)n^4A_{n1} + 3n^2p_{n2}^2[-b_0A_{n2} + a_0C_{n2}]\}}{n^4[4R^2(\lambda-\lambda_1)^2 - 9(a_0^2 + b_0^2)]}, \tag{22}$$

$$a_{n2} = (-8R\lambda)\frac{\{2R(\lambda-\lambda_1)n^4A_{n2} + 3n^2p_{n1}^2[-b_0A_{n1} + a_0C_{n1}]\}}{n^4[4R^2(\lambda-\lambda_1)^2 - 9(a_0^2 + b_0^2)]}, \tag{23}$$

$$c_{n1} = (-8R\lambda)\frac{\{2R(\lambda-\lambda_1)n^4C_{n1} + 3n^2p_{n2}^2[b_0C_{n2} + a_0A_{n2}]\}}{n^4[4R^2(\lambda-\lambda_1)^2 - 9(a_0^2 + b_0^2)]}, \tag{24}$$

$$c_{n2} = (-8R\lambda)\frac{\{2R(\lambda-\lambda_1)n^4C_{n2} + 3n^2p_{n1}^2[b_0C_{n1} + a_0A_{n1}]\}}{n^4[4R^2(\lambda-\lambda_1)^2 - 9(a_0^2 + b_0^2)]}. \tag{25}$$

The equivalent set of relations for $b_{n1,2}$, $d_{n1,2}$ is obtained from the above by replacing $a_{n1,2}$, $c_{n1,2}$ with $b_{n1,2}$, $d_{n1,2}$ respectively as well as $A_{n1,2}$, $C_{n1,2}$ with $B_{n1,2}$, $D_{n1,2}$ respectively. These $8m$ relationships may now be substituted into eqns (16) and (17), thereby reducing the set of equilibrium equations to only two equations in a_0 and b_0. Doing so, yields

$$-8R\lambda A_0-4R(\lambda-\lambda_1)a_0+3(8R\lambda)^2\frac{[4R^2(\lambda-\lambda_1)^2-9(b_0^2-a_0^2)]}{[4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)]^2}S_3$$
$$+18R(\lambda-\lambda_1)\frac{(8R\lambda)^2a_0}{[4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)]^2}S_1$$
$$-\frac{54(8R\lambda)^2a_0b_0}{[4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)]^2}S_2=0 \tag{26}$$

and

$$-8R\lambda B_0-4R(\lambda-\lambda_1)b_0-3(8R\lambda)^2\frac{[4R^2(\lambda-\lambda_1)^2+9(b_0^2-a_0^2)]}{[4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)]^2}S_2$$
$$+\frac{18R(\lambda-\lambda_1)(8R\lambda)^2b_0}{[4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)]^2}S_1+\frac{54(8R\lambda)^2a_0b_0}{[4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)]^2}S_3=0, \tag{27}$$

where

$$S_1=\sum_{n=1}^{m}\frac{1}{p_0^2}\{p_{n1}^2(A_{n1}^2+B_{n1}^2+C_{n1}^2+D_{n1}^2)+p_{n2}^2(A_{n2}^2+B_{n2}^2+C_{n2}^2+D_{n2}^2)\}, \tag{28}$$

$$S_2=\sum_{n=1}^{m}\frac{1}{p_0^2}\{n^2(A_{n1}A_{n2}+B_{n1}B_{n2}-C_{n1}C_{n2}-D_{n1}D_{n2})\}, \tag{29}$$

$$S_3=\sum_{n=1}^{m}\frac{1}{p_0^2}\{n^2(A_{n1}C_{n2}+A_{n2}C_{n1}+B_{n1}D_{n2}+B_{n2}D_{n1})\}. \tag{30}$$

The interesting feature of the above equations is that the imperfection parameters associated with the non-axisymmetric modes, $A_{n1,2}$, $B_{n1,2}\ldots$, appear in three and only three distinct summations. Therefore, the behaviour of the system depends on the value of these summations and not on the individual imperfection parameters.

CRITICAL STATES OF EQUILIBRIUM

The equilibrium states of the system are defined to be stable if the second variation of the energy is greater than zero. A critical state occurs when the second variation becomes zero. In the present analysis, the second variation is limited to small variations of the eigenfunctions eqns (10)–(12). This is equivalent to allowing variations of a_0, b_0, $a_{n1,2},\ldots$. The stability matrix is thus obtained from the potential energy expression, equation (13), where the components of this matrix have the form $\partial^2P^\lambda[u_i]/\partial z_1\partial z_2$, and where z_1, z_2 range over a_0, b_0, $a_{n1,2},\ldots$. The required matrix is of dimension $2+8m$. It is of a relatively simple form in that the first two rows and columns are full while the remainder is composed of 4×4 sub-matrices which lie along the diagonal. In addition, the two full rows and columns depend only on the non-axisymmetric deflection components; whereas, the 4×4 submatrices are dependent only on the axisymmetric deflection components. For demonstration purposes, the first six rows and columns are shown in (31).

$$\begin{matrix}
-4R(\lambda-\lambda_1)p_0^2 & 0 & 3n^2c_{n2} & 3n^2a_{n2} & 3n^2c_{n1} & 3n^2a_{n1} & \cdot\cdot \\
0 & -4R(\lambda-\lambda_1)p_0^2 & -3n^2a_{n2} & 3n^2c_{n2} & -3n^2a_{n1} & 3n^2c_{n1} & \cdot\cdot \\
3n^2c_{n2} & -3n^2a_{n2} & -2R(\lambda-\lambda_1)p_{n1}^2 & 0 & -3n^2b_0 & 3n^2a_0 & \cdot\cdot \\
3n^2a_{n2} & 3n^2c_{n2} & 0 & -2R(\lambda-\lambda_1)p_{n1}^2 & 3n^2a_0 & 3n^2b_0 & \cdot\cdot \\
3n^2c_{n1} & -3n^2a_{n1} & -3n^2b_0 & 3n^2a_0 & -2R(\lambda-\lambda_1)p_{n2}^2 & 0 & \cdot\cdot \\
3n^2a_{n1} & 3n^2c_{n1} & 3n^2a_0 & 3n^2b_0 & 0 & -2R(\lambda-\lambda_1)p_{n2}^2 & \cdot\cdot \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots &
\end{matrix} \tag{31}$$

In the matrix (31) the next four rows and columns demonstrate the influence of $b_{n1,2}$, $d_{n1,2}$. Also, n takes the values 1 to m. The system is defined as stable if the matrix is positive definite which requires all of the leading principal minors to be positive. The matrix in question is of dimension $2+8m$, and the principal minors are evaluated in the most concise manner by starting in the lower right-hand corner. If the principal minors are designated as M_i, $i = 1, 2+8m$, then they are

$$\begin{aligned} &M_1 = -2R(\lambda - \lambda_1)p^2_{m2}, \\ &M_2 = [2R(\lambda - \lambda_1)p^2_{m2}]^2, \\ &M_3 = -2R(\lambda - \lambda_1)p^2_{m2}m^4[4R^2(\lambda - \lambda_1)^2 - 9(b_0^2 + a_0^2)], \\ &M_4 = m^8[4R^2(\lambda - \lambda_1)^2 - 9(b_0^2 + a_0^2)]^2, \\ &M_5 = M_4 \times M_1, \quad M_6 = M_4 \times M_2, \quad M_7 = M_4 \times M_3, \quad M_8 \times M_4 \times M_4. \end{aligned} \tag{32}$$

The remainder of the first $8m$ principal leading principal minors are given by the following set of recurrence relations.

$$\begin{aligned} &M_{8j+1} = M_{8j} \times [-2R(\lambda - \lambda_1)p^2_{(m-j)2}], \\ &M_{8j+2} = M_{8j} \times [-2R(\lambda - \lambda_1)p^2_{(m-j)2}]^2, \\ &M_{8j+3} = M_{8j} \times \{-2R(\lambda - \lambda_1)p^2_{(m-j)2}(m-j)^4[4R^2(\lambda - \lambda_1)^2 - 9(b_0^2 + a_0^2)]\}, \\ &M_{8j+4} = M_{8j} \times (m-j)^8[4R^2(\lambda - \lambda_1)^2 - 9(b_0^2 + a_0^2)]^2, \\ &M_{8j+5} = M_{8j+4} \times M_{8j+1}/M_{8j}, \\ &M_{8j+6} = M_{8j+4} \times M_{8j+2}/M_{8j}, \\ &M_{8j+7} = M_{8j+4} \times M_{8j+3}/M_{8j}, \\ &M_{8j+8} = M_{8j+4} \times M_{8j+4}/M_{8j}, \end{aligned} \tag{33}$$

where $j = 1, m-1$.

The last two leading principal minors are more difficult to evaluate, but after a little manipulation and after the elimination of $a_{n1,2}$, $b_{n1,2}, \ldots$ they take the form

$$\begin{aligned} M_{8m+1} = {} & [-4R(\lambda - \lambda_1)p_0^2] \times M_{8m} \\ & \times \left\{1 - \frac{9(8R\lambda)^2}{2R(\lambda - \lambda_1)[4R^2(\lambda - \lambda_1)^2 - 9(a_0^2 + b_0^2)]^3} \right. \\ & \times \{R(\lambda - \lambda_1)S_1[4R^2(\lambda - \lambda_1)^2 - 9(b_0^2 + a_0^2) + 36b_0^2] \\ & + 3a_0S_3[4R^3(\lambda - \lambda_1)^2 - 9(b_0^2 + a_0^2) + 36b_0^2] \\ & \left. - 9b_0S_2[4R^2(\lambda - \lambda_1)^2 - 9(b_0^2 + a_0^2) + 12b_0^2]\}\right\}, \end{aligned} \tag{34}$$

and

$$\begin{aligned} M_{8m+2} = {} & [-4R(\lambda - \lambda_1)p_0^2]^2 \times M_{8m} \\ & \times \left[\left\{1 - \frac{9(8R\lambda)^2}{2R(\lambda - \lambda_1)[4R^2(\lambda - \lambda_1)^2 - 9(b_0^2 + a_0^2)]^3} \right.\right. \\ & \times \{R(\lambda - \lambda_1)S_1[4R^2(\lambda - \lambda_1)^2 - 9(b_0^2 + a_0^2) + 36a_0^2] \\ & + 9a_0S_3[4R^2(\lambda - \lambda_1)^2 - 9(b_0^2 + a_0^2) + 12a_0^2] \\ & \left. - 3b_0S_2[4R^2(\lambda - \lambda_1)^2 - 9(b_0^2 + a_0^2) + 36a_0^2]\}\right\} \\ & \times \left\{1 - \frac{9(8R\lambda)^2}{2R(\lambda - \lambda_1)[4R^2(\lambda - \lambda_1)^2 - 9(b_0^2 + a_0^2)]^3} \right. \\ & \times \{R(\lambda - \lambda_1)S_1[4R^2(\lambda - \lambda_1)^2 - 9(b_0^2 + a_0^2) + 36b_0^2] \end{aligned}$$

$$+3a_0S_3[4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)+36b_0^2]$$
$$-9b_0S_2[4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)+12b_0^2]\}\Big\}$$
$$-\left\{\frac{27(8R\lambda)^2}{2R(\lambda-\lambda_1)[4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)]^3}\right.$$
$$\times\{a_0b_012R(\lambda-\lambda_1)S_1-a_0S_2[4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)+36b_0^2]$$
$$\left.+b_0S_3[4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)+36a_0^2]\}\right\}^2\Bigg]. \tag{35}$$

It is now possible to determine the critical load for arbitrary values of A_0, B_0, $A_{n1,2},\ldots$, in a straight-forward manner. This involves the numerical solution of eqns (26) and (27) in conjunction with the requirement that

$$M_i>0,\quad i=1,2+8m \tag{36}$$

as the load parameter, λ, is increased from zero. The largest value of λ, called λ_c, which is such that all λ satisfying

$$\lambda_c\geq\lambda\geq 0 \tag{37}$$

also yields an equilibrium state which satisfies (36) is then defined as the critical load. This will, of course, include limit point and bifurcation critical states.

BIFURCATION STATES

In the present problem, it is possible to distinguish two types of bifurcation states; namely, primary and secondary. Primary bifurcation is defined to be a bifurcation state which occurs when all of the deflection parameters a_0, b_0, $a_{n1,2},\ldots$ are trivial. On the other hand, secondary bifurcation states are those which occur when any of the deflection parameters are non-zero.

(i) *Primary bifurcation*

Primary bifurcation represents the classical buckling situation and arises when $A_0=B_0=A_{n1,2}=\ldots=0$. The bifurcation load is therefore the classical critical load, λ_1.

(ii) *Secondary bifurcation*

It is possible to distinguish two secondary bifurcation cases. The simplest case occurs when A_0, B_0 are non-zero and the remaining imperfection parameters are zero. This combination of imperfection parameters, which leads to bifurcation from axisymmetric to combined axisymmetric and non-axisymmetric modes, was the case treated by Koiter[1]. The equilibrium equations are

$$-8R\lambda p_0^2A_0-4R(\lambda-\lambda_1)p_0^2a_0=0$$

and

$$-8R\lambda p_0^2B_0-4R(\lambda-\lambda_1)p_0^2b_0=0 \tag{38}$$

while $a_{n1,2}=b_{n1,2}=\ldots=0$ provided that the critical state, defined by

$$4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)=0 \tag{39}$$

is not attained. The bifurcation load is the least value of λ which satisfies equations (38) and (39). After normalization with respect to λ_1, the bifurcation load is found as

$$\lambda_B=1+3/2\left(\frac{S_0}{R^2\lambda_1^2}\right)^{1/2}-\left[3\left(\frac{S_0}{R^2\lambda_1^2}\right)^{1/2}+9/4\left(\frac{S_0}{R^2\lambda_1^2}\right)\right]^{1/2} \tag{40}$$

where

$$S_0=B_0^2+A_0^2. \tag{41}$$

The other case of secondary bifurcation results when S_1 is non-zero and S_0, S_2, S_3 are zero. The pertinent equilibrium equations are

$$-4R(\lambda-\lambda_1)a_0+\frac{18R(\lambda-\lambda_1)(8R\lambda)^2a_0}{[4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)]^2}S_1=0,$$

and (42)

$$-4R(\lambda-\lambda_1)b_0+\frac{18R(\lambda-\lambda_1)(8R\lambda)^2b_0}{[4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)]^2}S_1=0,$$

while $a_{n1,2}$, $b_{n1,2}$, ... are in general non-zero. Among the possible solutions of eqns (42), those of interest are

$$a_0=b_0=0 \tag{43}$$

for λ small and the solution of

$$-4R(\lambda-\lambda_1)+\frac{18R(\lambda-\lambda_1)(8R\lambda)^2}{[4R^2(\lambda-\lambda_1)^2-9(b_0^2+a_0^2)]^2}S_1=0, \tag{44}$$

for some larger value of λ.

The critical state of equilibrium is determined from M_{8m+1} which yields the condition

$$[4R^2(\lambda-\lambda_1)^2]^2-9/2(8R\lambda)^2S_1=0 \tag{45}$$

which in turn yields the bifurcation load. After normalization with respect to λ_1, it becomes

$$\lambda_B=1+\frac{3}{\sqrt{2}}\left(\frac{S_1}{R^2\lambda_1^2}\right)^{1/2}-\left[\frac{6}{\sqrt{2}}\left(\frac{S_1}{R^2\lambda_1^2}\right)^{1/2}+9/2\left(\frac{S_1}{R^2\lambda_1^2}\right)\right]^{1/2}. \tag{46}$$

At this load the system bifurcates from non-axisymmetric to combined non-axisymmetric and axisymmetric modes. This conclusion is obtained by noting that when $a_0=b_0=0$, then eqn (44) reduces to the stability criterion (eqn 45).

An interesting feature of the secondary bifurcation cases is that the bifurcation loads are identical in form. In fact, if $S_0=2S_1$, then the bifurcation loads are equal in magnitude.

LIMIT POINT CRITICAL LOADS

Limit point critical loads do not yield to analytical treatment and thus a numerical procedure is necessary. In the numerical work the factors A_0, B_0, S_1, S_2 and S_3 where chosen as

$$A_0=\epsilon_0\frac{h}{2\sqrt{2}},\quad B_0=\epsilon_0\frac{h}{2\sqrt{2}},$$

(47)

$$S_1=\epsilon_1^2\frac{h^2}{4},\quad S_2=\pm\epsilon_2^2\frac{h^2}{4},\quad S_3=\pm\epsilon_3^2\frac{h^2}{4}.$$

A_0 and B_0 have been chosen in the above manner in order that the bifurcation results of eqn (40) be identical to Koiter's bifurcation results. Furthermore, S_1, S_2 and S_3 have been chosen such that $S_0=S_1=\pm S_2=\pm S_3$ when $\epsilon_0=\epsilon_1=\epsilon_2=\epsilon_3$. The non-dimensional ϵ-parameters are not simple imperfection amplitude-shell thickness ratios, but may involve both the axial and circumferencial wave numbers. These parameters will be discussed later. Also, the choice of A_0 and B_0 positive in no way detracts from the generality of the results. For, negative values of A_0 and/or B_0 will lead to results which are identical to those of positive A_0 and B_0 if the appropriate signs are taken for S_2 and S_3. Therefore, since all combinations of sign are taken for S_2 and S_3, the possibilities have been exhausted.

DISCUSSION OF RESULTS

Typical results of the numerical calculations are shown in Figs. 1–4. The first of these considers the overall influence of various combinations of the imperfection components and, in particular, draws attention to the important role of the non-axisymmetric imperfection components. The subsequent figures investigate a more local type of interaction between the various imperfection components. Figures 2a,b deal with the imperfection components S_0 and S_1 which individually lead to secondary bifurcation states but which, when combined, lead to limit

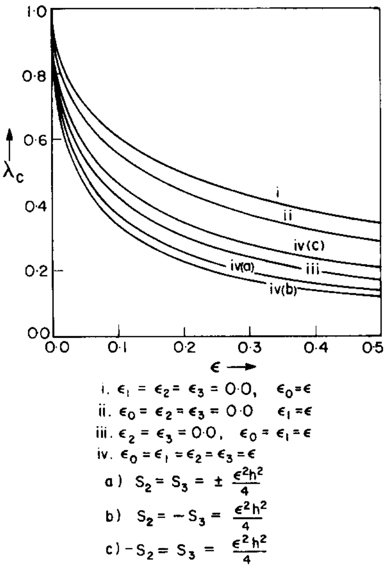

Fig. 1. Overall influence of the imperfection parameters.

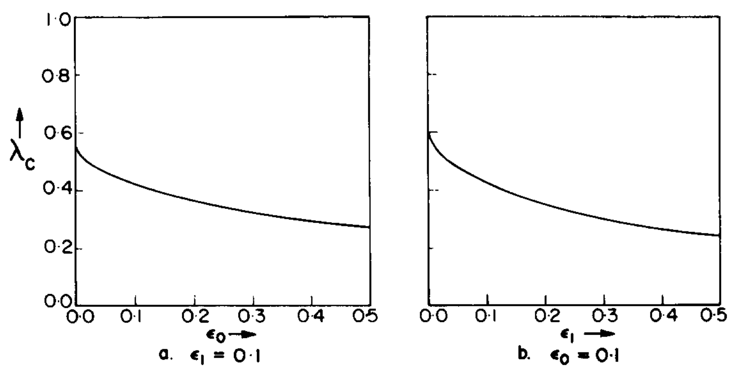

Fig. 2. Interaction between S_0 and S_1.

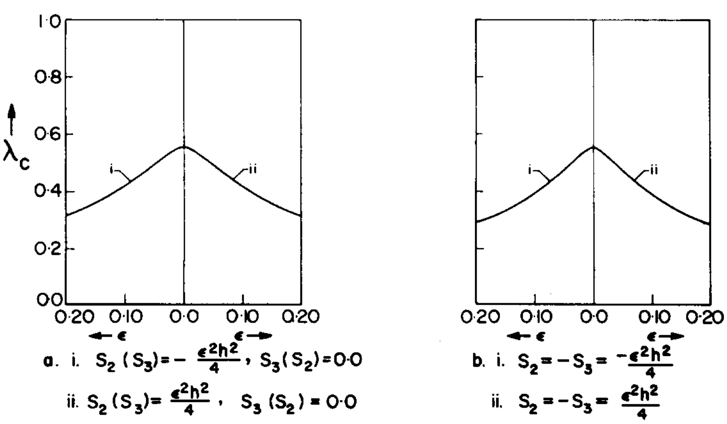

Fig. 3. Local influence of the non-axisymmetric imperfection components S_1, S_2 and S_3: $\epsilon_0 = 0{\cdot}0$, $\epsilon_1 = 0{\cdot}1$.

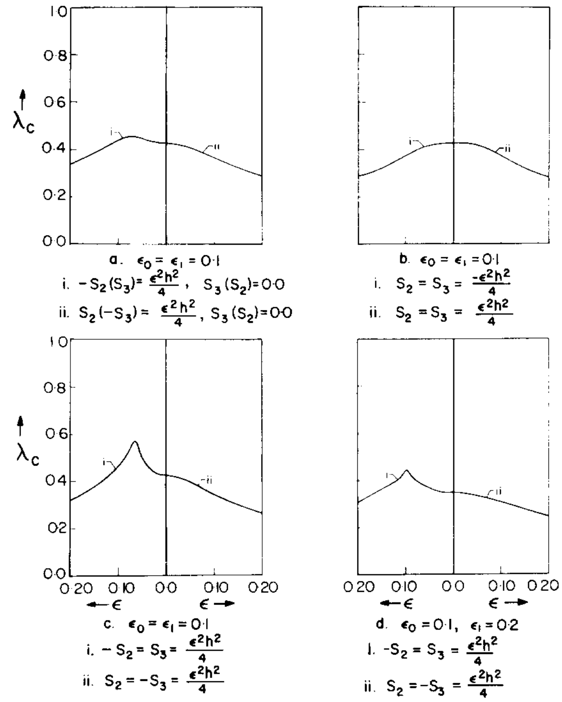

Fig. 4. Local interaction between the axisymmetric (S_0) and the non-axisymmetric (S_1, S_2, S_3) imperfection components.

point states. As is expected, a vertical tangent is evident in the critical load-initial imperfection curves as either S_0 or S_1 approaches zero. This feature emphasises the importance of secondary bifurcation as well as the interaction between axisymmetric and non-axisymmetric imperfection components. Figures 3a,b are devoted to the investigation of the non-axisymmetric imperfection components. These figures indicate that the addition of either $\pm S_2$ or $\pm S_3$, to the system containing only S_1, have identical influences on the critical load. The remaining figures demonstrate the influence of various combinations of S_2 and S_3 when certain components of S_0 and S_1 are present in the system. Since the basic state associated with S_0 and S_1 is a limit point, the addition of S_2 and S_3 does not lead to drastic changes in the critical load. Further, the slope of the critical load-imperfection curves is apparently continuous and equal to zero when ϵ is zero. One interesting feature of the present results is evident in Figs. 4c,d in which a peak is found. This is due to an interaction between the different imperfection components which in essence cancel each other.

The results, as presented, yield relationships between the critical load and ϵ_0, ϵ_1, ϵ_2 and ϵ_3. This does not yield any information as to the relationship between ϵ_0, ϵ_1, ϵ_2, ϵ_3 and the amplitudes of particular modes and/or the wavelengths of these modes. For illustration purposes, it is convenient to assume an initial imperfection of the form

$$w_0(x, \theta) = \mu_0 h \sin(p_0 x - \pi/2) + \mu_{n1} h \sin p_{n1} x \cos n\theta + \mu_{n2} h \sin p_{n2} x \cos n\theta$$

where μ_0, $\mu_{n1,2}$ are the amplitudes of the modal imperfections, normalized with respect to the thickness of the shell.

The relationships between μ_0, μ_{n1}, μ_{n2} and ϵ_0, ϵ_1, ϵ_2, ϵ_3 are therefore

$$\epsilon_0 = \mu_0$$

$$\epsilon_1 = \frac{1}{2p_0}[p_{n1}^2 \mu_{n1}^2 + p_{n2}^2 \mu_{n2}^2]^{1/2}$$

$$|\epsilon_2| = \frac{n}{2p_0}[|\mu_{n1}\mu_{n2}|]^{1/2}$$

$$|\epsilon_3| = 0.$$

Since $p_{n1,2}$ and n are always less than p_0, the above demonstrates that the influence of the non-axisymmetric imperfection amplitudes $\mu_{n1,2}$ will be suppressed in relation to the axisymmetric one, μ_0. This suppression is directly related to the wave-lengths, both axial and circumferential, of the imperfections. For example, in the particular case that μ_{n2} vanishes and μ_{n1} is equal to μ_0,

$$\epsilon_1 = \frac{p_{n1}}{2p_0}\epsilon_0$$

which reduces to

$$\epsilon_1 < 1/2\epsilon_0$$

for any admissible value of n, R/h and ν.

On the other hand, it must be emphasised that ϵ_1, ϵ_2, ϵ_3 are composed of summations involving $8m$, $4m$ and $4m$ imperfection components respectively, ($m = 9$ if $R/h = 100$ and $\nu = 0{\cdot}3$ and increase with larger R/h ratios). Therefore, it is not justified to ignore the non-axisymmetric imperfections, and it could easily occur that they will be predominant.

Another point of interest deals with the assumption of infinite length in the shell. Basically, the coincidence of the $2+8m$ critical loads (eigenvalues) depends on p being continuous which requires the shell to be infinitely long. Therefore, if the shell is assumed to be of finite length the problem goes from one of coincident to one of nearly coincident critical loads. This near coincidence results in added analytical difficulties but it must be expected that the imperfection sensitivity will not be altered appreciably.

In summary, it has been demonstrated that the inclusion of non-axisymmetric imperfections can lead to intermodel behaviour which results in severe reductions of the critical load. Bifurcation and limit point critical loads have been examined and in this examination it must be concluded that both axisymmetric and non-axisymmetric imperfections play equally important roles.

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